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Mass transfer between a two-dimensional wall jet with a heterogeneous chemical reaction of the first order and a wall: analytical solution

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Abstract—The work studies mass transfer in a two-dimensional submerged wall jet flow with a chemical reaction of the first order at the surface. The approach suggested previously by Apelblat [1] for the analysis of a mass transfer with a first-order chemical reaction at the interface in a boundary layer flow is generalized for the case of a submerged two-dimensional wall jet flow. The solution of the problem is obtained in a closed analytical form. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Mass transfer with a heterogeneous chemical reaction of the first order arises in a heterogeneous catalysis, chemical absorption and chemical engineering. For comparison between theory and experimental data, simple and exact solutions of the problem are desirable. At the same time, exact analytical solutions were derived only for very simple geometry and flow configurations.

An analytical solution of a mass transfer problem coupled with an irreversible chemical reaction of the first order at a surface for a flow with constant velocity (plug flow) is obtained in refs. [1, 2]. Apelblat [1] derived exact analytical solutions of a mass transfer problem with a heterogeneous chemical reaction of the first-order in a flow with a constant velocity gradient at the interface (Couette flow) and in a flow with a moving interface (generalized Couette flow). In his further study [3] Apelblat analyzed the effect of molecular diffusion in the direction of convective transport. Diffusion with interfacial chemical reaction in a laminar channel flow is investigated by Cowherd and Haelscher [4]. Ghez [5] considered mass transport in a multicomponent system with a surface chemical reaction of the first order. In refs. [1, 2, 6] mass transfer with a first-order chemical reaction at the surface between a flat plate and a parallel fluid flow is studied analytically by three different methods. The boundary layer problem is the most complicated one from the mathematical point of view. Formulation of a convective heat transfer problem with mixed boundary conditions has the same mathematical form as a convective mass transfer problem with a heterogeneous first-order chemical reaction. The mixed boundary conditions in the problems of mass or heat transfer also arise when mass or heat transfer through an interface is inhibited by the presence of surfactants [6]. Complex mass transfer between a plane jet and a wall arises in heterogeneous catalysis, in chemical engineering operations, in etching, etc.

The wall jet forms at the trailing edge of a gas slug in gas–liquid slug flow [7] when a semi-infinite jet emerges from the thin circular slot between a gas slug and a tube wall and spreads in a liquid plug along the tube wall. The solution of a mass or heat transfer problem between a jet and a wall can be used for the analysis of mass or heat transfer between gas–liquid slug flow and a wall with mixed boundary conditions. The solution of a heat transfer problem between a wall and gas–liquid slug flow with Dirichlet and Neumann boundary conditions was derived in ref. [8].

FORMULATION OF THE PROBLEM

Consider mass transfer between a solid surface and an adjacent laminar wall jet flow, whereby a jet emerges from a thin slot and spreads along the surface (see Fig. 1). The soluble substance with concentration $c(x, y)$ flows with a fluid and is dispersed under the combined effects of diffusion and chemical reaction. In diffusion kinetics, mass flux is usually directed from a liquid with initial concentration c_0 to a solid surface. The hydrodynamics of a wall jet flow was studied in refs. [9, 10] where the following formulas for streamline function, velocity components and wall shear stress were derived:

$$\psi = (\nu E x)^{1/4} F(\eta) \quad \eta = \left(\frac{E}{\nu^3}\right)^{1/4}_{\xi} \frac{y}{x^{3/4}} \tag{1}$$

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \tag{2}$$

NOMENCLATURE

- $B(z,w)$ beta function
- $C = c/c_0 - 1$
- $c(x, y)$ molar concentration at point x, y
- c_0 concentration in the bulk of liquid
- D diffusion coefficient
- $J_\nu(z)$ Bessel function of the first kind of order ν
- q flux at the interface
- k rate constant of surface chemical reaction
- $U(a, b, z)$ Kummer function
- x coordinate in the direction parallel to the interface
- $\tilde{x}$ dimensionless coordinate
- y coordinate in the direction normal to the interface
- u velocity component in x-direction
- v velocity component in y-direction
- X transformed variable of x
- Y transformed variable of y
- $F(\eta)$ function, equation (4)
- Q flow rate of liquid
- J total jet momentum
- E constant, equation (6)
- Re Reynolds number
- Da Damkohler number
- $Sc = \nu/D$ Shmidt number
- $W_{l,m}(z)$ Whittaker function.

Greek symbols

- $\alpha, \beta, \gamma, \lambda$ constants
- $\Gamma(z)$ gamma function
- η similarity variable
- ν kinematic viscosity, constant
- μ dynamic viscosity, constant
- ρ fluid density, constant
- ϕ variable
- ψ stream function
- τ wall shear stress.

$$\tau = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0.221\mu\left(\frac{E^3}{\nu^5 x^5}\right)^{1/4}. \quad (3)$$

The unknown function $F(\eta)$ in formula (1) is determined from the solution of an ordinary differential equation

$$4F''' + FF'' + 2F'^2 = 0 \quad (4)$$

where the prime denotes derivative with respect to η. The invariant E in equations (1) and (3) is determined by the following formulas:

$$E = \int_0^\infty \psi u^2 \, \mathrm{d}y = \text{const}, \quad (5)$$

or

$$E = \frac{9QJ}{20 \cdot \rho}, \quad (6)$$

where

$$Q = \int_0^\infty u \, \mathrm{d}y, \quad J = \rho \int_0^\infty u^2 \, \mathrm{d}y. \quad (7)$$

In equation (7), Q is a fluid flow rate and J is the momentum flux density.

At a steady state, neglecting diffusion in the direction of convective transport, the mass transfer is governed by the equation of convective diffusion

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\frac{\partial^2 c}{\partial y^2} \quad (8)$$

with initial and boundary conditions

$$c = c_0 \quad \text{for } x = 0 \quad \text{and} \quad y > 0 \quad (9)$$

$$c = c_0 \quad \text{for } x > 0 \quad \text{and} \quad y \to \infty \quad (10)$$

$$D\frac{\partial c}{\partial y} = kc \quad \text{for } x > 0 \quad \text{and} \quad y = 0. \quad (11)$$

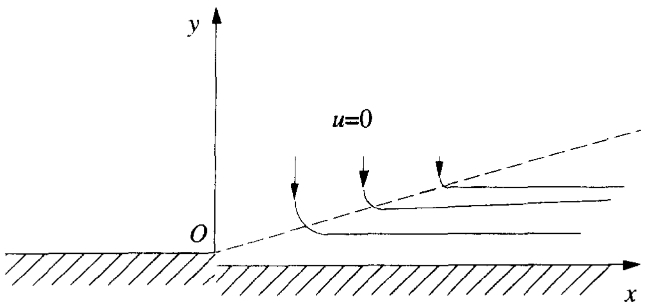

Fig. 1. Scheme of a wall jet flow.

SOLUTION OF THE PROBLEM

The above boundary value problem (8)–(11), with mixed boundary conditions, is solved in the approximation of a thin concentration boundary layer. In a case of large Schmidt numbers $Sc = \nu/D \gg 1$, the thickness of the concentration boundary layer is considerably less than that of the viscous boundary layer. Then it is reasonable to assume that the dependence of the longitudinal velocity component upon y is linear, i.e.

$$u = \frac{\tau}{\mu}y \tag{12}$$

where τ is determined by equation (3). The transversal velocity component is found from the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{13}$$

Introduce a new variable

$$\psi = \int_0^y u\,\mathrm{d}y' = \frac{0.221E^{3/4}x^{-5/4}y^2}{2\nu^{5/4}}. \tag{14}$$

Then equation (8) can be reduced to

$$\frac{\partial c}{\partial x} = D\frac{\partial c}{\partial \psi}\left(u\frac{\partial c}{\partial \psi}\right). \tag{15}$$

Taking into account that

$$\frac{u}{\sqrt{\psi}} = \frac{(0.221)^{1/2}E^{3/8}x^{-5/8}\cdot 2^{1/2}}{\nu^{5/8}} \tag{16}$$

equation (15) yields

$$\frac{\partial c}{\partial x}x^{5/8} = \alpha\frac{\partial}{\partial \psi}\left(\sqrt{\psi}\frac{\partial c}{\partial \psi}\right) \tag{17}$$

where

$$\alpha = \frac{D(0.221)^{1/2}E^{3/8}\cdot 2^{1/2}}{\nu^{5/8}}. \tag{18}$$

Introduce the new variables

$$X = \alpha\cdot\tfrac{8}{3}x^{3/8} \quad Y = 2^{2/3}\sqrt{\psi}. \tag{19}$$

Equation (17) in these new variables can be rewritten as

$$Y\frac{\partial c}{\partial X} = \frac{\partial^2 c}{\partial Y^2}. \tag{20}$$

The boundary conditions (9)–(11) in the new variables read

$$c = c_0 \quad \text{for } X = 0 \quad \text{and} \quad Y = 0 \tag{21}$$

$$c = c_0 \quad \text{for } X > 0 \quad \text{and} \quad Y \to \infty \tag{22}$$

$$\frac{\partial c}{\partial Y} = \beta X^{5/3}c, \tag{23}$$

where

$$\beta = \frac{k\cdot 2^{1/3}3^{5/3}}{\alpha^{8/3}\cdot 2^5} = \frac{3^{5/3}k\nu^{5/3}E^{-1}}{2^6D^{8/3}(0.221)^{4/3}}. \tag{24}$$

Using the new variable $C(X, Y) = c(X, Y) - c_0$ the above boundary value problem can be formulated as follows:

$$Y\frac{\partial C}{\partial X} = \frac{\partial^2 C}{\partial Y^2} \tag{25}$$

$$C = 0 \quad \text{at } X = 0 \quad \text{and} \quad Y = 0 \tag{26}$$

$$C = 0 \quad \text{at } X > 0 \quad \text{and} \quad Y \to \infty \tag{27}$$

$$\frac{\partial C}{\partial Y} = \beta X^{5/3}C + \beta X^{5/3}c_0 \quad \text{for } Y = 0. \tag{28}$$

Applying the Laplace transform to equation (25), using the boundary conditions (26) and (27), and using the technique developed in ref. [1], we obtain

$$C(X, Y) = \int_0^X G(t)\int_0^\infty \frac{Y^{1/2}u^{1/6}}{2\sqrt{3}\pi} \times \exp[-u(X-t)]J_{1/3}(\phi)\,\mathrm{d}u\,\mathrm{d}t \tag{29}$$

where $\phi = 2u^{1/2}\cdot Y^{3/2}/3$, $J_{1/3}(\phi)$—Bessel function. The unknown function $G(X)$ is determined from the remaining boundary condition (28). Using equations (28) and (29) and taking into account that (see, e.g. [1])

$$\frac{\mathrm{d}}{\mathrm{d}Y}[Y^{1/2}J_{1/3}(\phi)]_{Y=0} = \frac{3^{2/3}u^{1/6}}{\Gamma(1/3)} \tag{30}$$

the following integral Abel equation for evaluating $G(X)$ is derived:

$$\int_0^X \frac{G(t)}{(X-t)^{4/3}} = \gamma X^{5/3} \tag{31}$$

where

$$\gamma = 3^{5/6}\cdot 2\pi\beta. \tag{32}$$

The solution of this Abel equation reads (for details see ref. [11])

$$G(X) = -\frac{5\gamma}{2\pi\sqrt{3}}\int_0^X t^{1/3}(X-t)^{2/3}\,\mathrm{d}t. \tag{33}$$

Introduce the new variable $\xi = t/X$. Then the integral in equation (33) can be expressed through the beta function (see, e.g. [12])

$$G(X) = -\frac{5\gamma X^2}{2\pi\sqrt{3}}\int_0^1 \xi^{1/3}(1-\xi)^{2/3}\,\mathrm{d}\xi = -\frac{5}{2\pi}\frac{X^2}{\sqrt{3}}B(\tfrac{4}{3}, \tfrac{5}{3}). \tag{34}$$

Using the known relation between the beta function and the gamma function (for details see [12]), we find

$$G(X) = -\frac{5\gamma\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{2\pi 3^{5/2}}X^2. \quad (35)$$

Equations (29) and (35) yield the following formula for concentration distribution:

$$\frac{c(X,Y)}{c_0} = 1-\lambda\cdot\int_0^X (X-t)^2\int_0^\infty Y^{1/2} \times e^{-uX}u^{1/6}J_{1/3}(\phi)\,du\,dt, \quad (36)$$

where

$$\lambda = \frac{5\gamma\Gamma(1/3)\Gamma(2/3)}{108\pi^2} = \frac{5k\Gamma(1/3)\Gamma(2/3)\nu^{5/3}}{128\pi\sqrt{3}(0.221)^{4/3}ED^{8/3}}. \quad (37)$$

The result can be reduced to Whittaker function using the following formulas (see, e.g. refs. [12, 13]):

$$\int_0^\infty e^{-uX}u^{1/6}J_{1/3}(\phi)\,du = \frac{Y^{1/2}}{3^{1/3}X^{4/3}}\exp\left(-\frac{Y^3}{9X}\right) \quad (38)$$

$$\int_0^X (X-t)^2 t^{-4/3}\exp\left(-\frac{Y^3}{9t}\right) = \frac{2\cdot 3^{4/3}X^{7/3}}{Y^2} \times \exp\left(-\frac{Y^3}{18X}\right)W_{-7/3,\,1/6}\left(\frac{Y^3}{9X}\right). \quad (39)$$

Thus, we arrived at the analytical expression for the distribution of concentration in the wall jet flow with the first-order surface chemical reaction

$$\frac{c(X,Y)}{c_0} = 1-\frac{6\lambda X^{7/3}}{Y}\exp\left(-\frac{Y^3}{18X}\right)W_{-7/3,\,1/6}\left(\frac{Y^3}{9X}\right). \quad (40)$$

The distribution of concentration can also be written in terms of the confluent hypergeometric Kummer function. Taking into account the relation between the Whittaker and Kummer functions (see, e.g. ref. [12])

$$W_{l,m}(z) = e^{-z/2}z^{1/2+m}U(\tfrac{1}{2}+m-l,1+2m,z), \quad (41)$$

the distribution of concentrations can be written as follows:

$$\frac{c(X,Y)}{c_0} = 1-2\cdot 3^{1/3}\lambda X^2\exp\left(-\frac{Y^3}{9X}\right)U\left(\frac{8}{3},\frac{2}{3},\frac{Y^3}{9X}\right). \quad (42)$$

Using the relation

$$U(a,b,0) = \frac{\pi}{\sin\pi b}\left(\frac{1}{\Gamma(1+a-b)\Gamma(b)}\right), \quad (43)$$

we obtain the explicit formula for the mass flux at the wall

$$q = kc|_{Y=0} = kc_0\left(1-\frac{3\pi\lambda\cdot X^2}{\Gamma(2/3)}\right). \quad (44)$$

Using the dimensionless variables

$$Re = \frac{Q}{\nu},\quad Da = \frac{k}{D}\frac{\nu^2}{J}\frac{\rho}{},\quad \tilde{x} = \frac{x}{\nu^2}\frac{J}{\rho},$$

formula (44) can be written as

$$\frac{q}{kc_0} = 1-1.939\frac{Da}{Re^{1/4}}\frac{\tilde{x}^{3/4}}{Sc^{1/3}}. \quad (45)$$

Dependence of the normalized mass flux q/kc_0 upon dimensionless coordinate $\tilde{x}$ as given by equation (45)

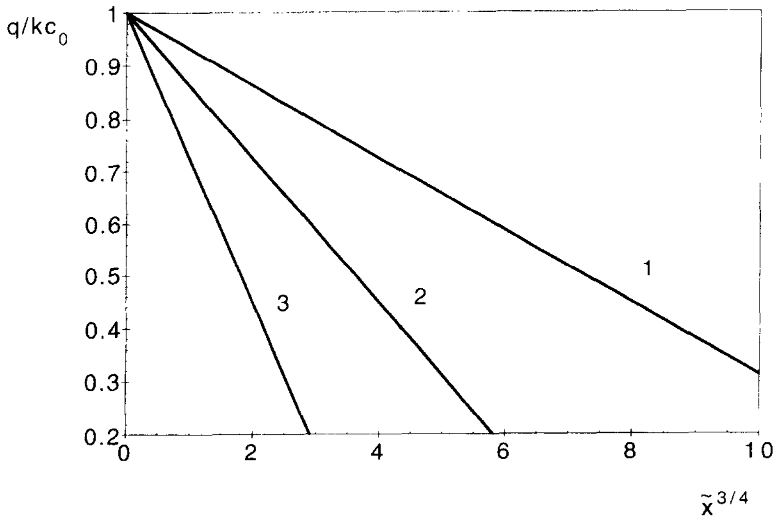

Fig. 2. Dependence of a normalized mass flux q/kc_0 upon dimensionless coordinate $\tilde{x}$. $Re = 10$. (1) $Da = 0.5$; (2) $Da = 1.0$; (3) $Da = 2.0$.

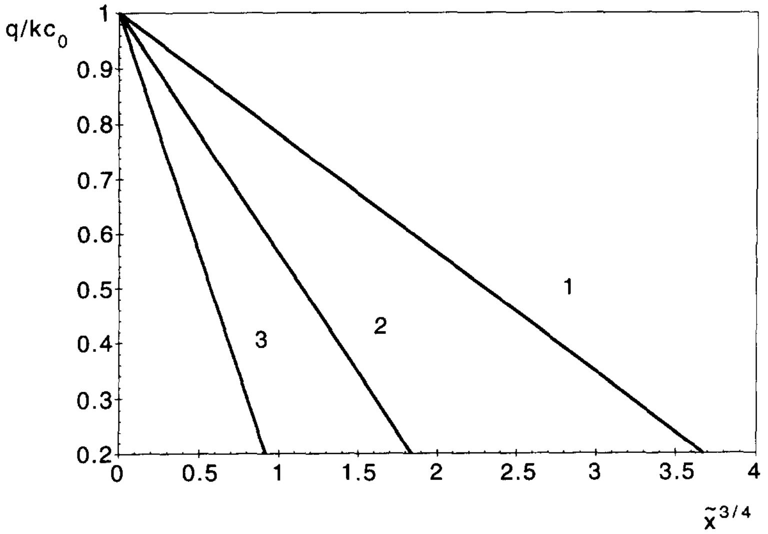

Fig. 3. Dependence of normalized mass flux q/kc_0 upon dimensionless coordinate $\tilde{x}$. $Re = 0.1$. (1) $Da = 0.5$; (2) $Da = 1.0$; (3) $Da = 2.0$.

is shown in Figs. 2 and 3 for $Sc = 500$, $Re = 0.1$ and 10 and $Da = 0.5$; 1.0 and 2.0.

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